

# ON THE NUCLEARITY OF DUAL GROUPS

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ABSTRACT. We prove that the dual space of a locally convex nuclear  $k_\omega$  vector space endowed with the compact–open topology is a locally convex nuclear vector space. An analogous result is shown for nuclear groups. As a consequence of this, we obtain that nuclear  $k_\omega$ –groups are strongly reflexive.

## 1. INTRODUCTION

Since Grothendieck defined nuclear vector spaces in the fifties ([9]), their properties, both in the context of the theory of vector spaces and their applications in analysis have been studied.

In the monograph [14] a survey is given about locally convex nuclear vector spaces. Several years later, in the nineties, Banaszczyk generalized the setting and introduced nuclear groups in [4]. He was able to show that nuclear groups share many properties of locally convex nuclear vector spaces and he was able to prove a version of the Bochner and the Lévy Theorem for nuclear groups. Even more, studying subgroups of locally convex nuclear vector spaces (which are typical examples for nuclear groups) he gave surprising characterizations of nuclear Fréchet spaces by means of their subgroups ([3]).

Nuclear groups have also good properties with respect to Pontryagin duality. Before recalling them, we remember the fundamental settings.

For an abelian topological group  $G$ , the set of continuous **characters** (i.e. homomorphisms in the compact circle group  $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ ) forms a group with multiplication defined pointwise. This group is denoted by  $G^\wedge$  and called **character group** or **dual group** of  $G$ . If  $G^\wedge$  is endowed with the compact–open topology, it is an abelian Hausdorff group, in particular a topological group. This permits to form the second dual group  $G^{\wedge\wedge} := (G^\wedge)^\wedge$ . A (necessarily abelian Hausdorff) topological group  $G$  is called **Pontryagin reflexive** if the canonical mapping

$$\alpha_G : G \rightarrow G^{\wedge\wedge}, \quad x \mapsto (\chi \mapsto \chi(x))$$

is a topological isomorphism.

These definitions imitate the dual space of a topological vector space and their duality theory. Fortunately, the two concept coincide, which means:

**Theorem 1.1.** *If  $E$  is a topological vector space, the mapping*

$$E'_{co} \rightarrow E^\wedge, \quad \varphi \mapsto e^{2\pi i \varphi}$$

*from the topological dual  $E'$  endowed with the compact–open topology into  $E^\wedge$  is a topological group isomorphism.*

A proof can be found in [18].

In the realm of topological vector spaces is an intensively studied object (recall that the strong topology on  $E'$ , a neighborhood basis of 0 is given by the polars of bounded subsets of  $E$ ). In [14] those nuclear space are characterized the strong dual of which is again a locally convex nuclear vector space.

Since we are interested in Pontryagin duality, the natural question which arises is to find sufficient conditions on a nuclear space  $E$  (resp. a nuclear group  $G$ ) such that  $E_{co}$  (resp.  $G^\wedge$ ) is a nuclear vector space.

One result in this direction in the realm of nuclear groups was achieved in [4] (16.1) and generalized in [2] (20.35) and (20.36):

**Theorem 1.2.** *If  $G$  is a metrizable nuclear group then  $G^\wedge$  is a nuclear group.*

From this it is easy to obtain

**Corollary 1.3.** *If  $E$  is a metrizable nuclear locally convex vector space then  $E'_{co}$  is a locally convex nuclear space.*

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One only has to consider the isomorphism  $E'_{co} \rightarrow E^\wedge$  and to recall that a topological vector space is a nuclear group if and only if it is a locally convex nuclear vector space ((8.9) in [4]).

The character group of a metrizable group is always a  $k_\omega$ -space ((4.7) in [2] or Theorem 1 in [7]) which means it has a countable cobasis for the compact sets and the topology is the final topology induced by the compact subsets.

Now it is natural to ask whether the dual group of a nuclear  $k_\omega$ -group is a nuclear group again. In 6.11 we will give an affirmative answer to this question. As a consequence of this it is not difficult to deduce that a nuclear  $k_\omega$ -group  $G$  is **strongly reflexive** which means that all closed subgroups and all Hausdorff quotient groups of  $G$  and  $G^\wedge$  are Pontryagin reflexive.

The article is organized as follows: In the second section we recall the definition of  $s$ -numbers. A typical example are the Kolmogoroff numbers which assign to two symmetric and convex subsets  $X$  and  $Y$  of a real vector space  $E$  a decreasing sequence  $(d_k(X, Y))$  of real numbers (or  $\infty$ ). These sequences measure how big  $X$  is w.r.t.  $Y$ . Drawing on results of [6], we show that  $d_k(X, Y)$  can be approximated by  $d_k(N \cap X, Y)$  for a suitable  $k$ -dimensional subspace  $N$ . This means: If  $X$  is rather big w.r.t.  $Y$  then already a finite-dimensional subset of  $X$  is rather big.

Based upon this result, we show in section 3 that a non-nuclear real Frechét space has a compact convex set  $K$  and a convex 0 neighborhood  $U$  such that  $d_k(K, U)$  is rather big, more precisely  $(k^3 d_k(K, U))$  is unbounded.

In section 4 we define nuclear vector groups (locally convex vector groups), which are roughly speaking locally convex nuclear vector spaces (locally convex vector spaces) over  $\mathbb{R}$  with the discrete topology. We give a similar characterization as above for metrizable locally convex vector groups which are not nuclear vector groups.

Afterwards, in section 5, we recall both the definition and a representation of nuclear groups. The latter allows us to find a null-sequence  $(g_n)$  and a neighborhood  $U$  in a non-nuclear metrizable and locally quasi-convex group  $G$  such that  $(d_k(\{\pm g_n \mid n \geq n_0\} \cup \{0\}, U)) \leq (ck^9)$  is impossible for every  $n_0 \in \mathbb{N}$  if the constant  $c$  is sufficiently small.

After these preparations, in section 6 we will first prove (based on the results of section 3) that the dual space  $E'_{co}$  of a locally convex nuclear vector space  $E$  which is a  $k_\omega$ -space, is again nuclear. For groups, an analogous result is true, however, the proof is more intricate and one more technical lemma is needed (in order to manipulate the constant  $c$  mentioned above).

Combining this result with well known results on nuclear groups and  $k_\omega$ -groups, we obtain that a nuclear  $k_\omega$ -group is strongly reflexive.

## 2. A PROPERTY OF KOLMOGOROFF NUMBERS

Let  $\mathcal{L}$  denote the class of all bounded linear operators between seminormed spaces. For seminormed spaces  $E$  and  $F$ , the set of bounded linear operators  $E \rightarrow F$  will be denoted by  $\mathcal{L}(E, F)$ . We recall the following definition:

**Definition 2.1.** A mapping  $s$  assigning to every  $\mathcal{L} \ni T : E \rightarrow F$  a sequence of real numbers  $(s_k(T))_{k \in \mathbb{N}}$  satisfying the properties (S 1) to (S 5) is called an  $s$ -**number**.

- (S1)  $\|T\| = s_1(T) \geq s_2(T) \geq \dots \geq 0$ ,
- (S2)  $s_k(S + T) \leq s_k(S) + \|T\|$  for all  $S, T \in \mathcal{L}(E, F)$  and for all  $k \in \mathbb{N}$ ,
- (S3)  $s_k(R \circ S \circ T) \leq \|R\| s_k(S) \|T\|$  for all  $T \in \mathcal{L}(E_0, E)$ ,  $S \in \mathcal{L}(E, F)$ ,  $R \in \mathcal{L}(F, F_0)$  and  $n \in \mathbb{N}$ ,
- (S4) If  $\dim T(E) < k$  then  $s_k(T) = 0$ ,
- (S5)  $s_k(I_k) = 1$  for all  $k \in \mathbb{N}$  where  $I_k : \mathbb{R}^k \rightarrow \mathbb{R}^k$  is the identity and  $\mathbb{R}^k$  is endowed with the euclidean norm.

**Remark 2.2.** Originally,  $s$ -numbers have been introduced in realm of bounded linear operators between Banach spaces ([13]).

Before presenting some examples, we introduce the following

**Notation 2.3.** For a subspace  $F$  of a seminormed space  $E$ ,

$$Q_F^E : E \rightarrow E/F$$

denotes the canonical projection and

$$U_E = \{x \in E \mid \|x\| \leq 1\}$$

the unit ball of  $E$ .

**Examples 2.4.** Let  $T : E \rightarrow F$  be a bounded linear operator between seminormed spaces.

(i) The **Kolmogoroff numbers**  $(d_k)$ , defined by

$$\begin{aligned} d_k(T) &:= \inf\{\|Q_L^F \circ T\| \mid \dim L < k\} \\ &= \inf\{\inf\{c > 0 \mid T(U_E) \subseteq cU_F + L\} \mid \dim L < k\} \\ &= \inf\{c > 0 \mid \exists L \leq E, \dim L < k, U_E \subseteq cU_F + L\}, \end{aligned}$$

are  $s$ -numbers. [Cf. 11.6 in [13].]

More generally, for symmetric and convex subsets  $X, Y$  in a vector space  $E$ , we put

$$d_k(X, Y) := \inf\{c > 0 \mid \exists L \leq E, \dim L < k, X \subseteq cY + L\} \in [0, \infty]$$

Then  $d_k(T) = d_k(T(U_E), U_F)$ .

(ii) The **Hilbert numbers**  $(h_k)$ , defined by

$$\begin{aligned} h_k(T) &:= \sup\{d_k(S_2 \circ T \circ S_1) \mid S_1 \in \mathcal{L}(H_1, E), S_2 \in \mathcal{L}(F, H_2), \\ &\quad \|S_1\| \leq 1, \|S_2\| \leq 1, H_1, H_2 \text{ Hilbert spaces}\}, \end{aligned}$$

are  $s$ -numbers. [Cf. 11.4 in [13]. The definition given in (11.4.1) in [13] of Hilbert numbers is different. However, the definitions coincide according to (11.3.4) in [13].]

The following Remark will clarify the connection between the two different settings of Kolmogoroff diameters.

**Remark 2.5.** Let  $X_1$  and  $X_2$  be symmetric and convex subsets of a vector space  $E$  with  $d_1(X_1, X_2) < \infty$ . Let  $p_i$  denote the Minkowski functional of  $X_i$  defined on  $\langle X_i \rangle$ . Then

$$d_k(X_1, X_2) = d_k(T) \quad \text{where} \quad T : (\langle X_1 \rangle, p_1) \rightarrow (\langle X_2 \rangle, p_2), \quad x \mapsto x.$$

Indeed, for every  $0 < \varepsilon < 1$  we have  $(1 - \varepsilon)B_{p_i} \subseteq X_i \subseteq B_{p_i}$  and hence

$$\begin{aligned} (1 - \varepsilon)d_k(B_{p_1}, B_{p_2}) &= d_k((1 - \varepsilon)B_{p_1}, B_{p_2}) \leq \\ &\leq d_k(X_1, X_2) \leq d_k(B_{p_1}, (1 - \varepsilon)B_{p_2}) \leq \frac{1}{1 - \varepsilon}d_k(B_{p_1}, B_{p_2}). \end{aligned}$$

We gather some simple properties of the Kolmogoroff numbers, which will be used frequently in the sequel.

**Lemma 2.6.** (i)  $d_{k+l-1}(X, Z) \leq d_k(X, Y) \cdot d_l(Y, Z)$  for all  $k, l \in \mathbb{N}$ .

(ii) If  $N$  is a subspace contained in  $X$  and if  $d_1(X, Y) < \infty$ , then  $d_k(X, Y) = d_k(Q_N^E(X), Q_N^E(Y))$ .

(iii) If  $N$  is a subspace contained in  $Y$ , then  $d_k(X, Y) = d_k(X + N, Y)$ .

(iv) For  $\lambda, \mu > 0$  we have  $d_k(\lambda X, \mu Y) = \frac{\lambda}{\mu}d_k(X, Y)$ .

(v)  $d_k(T^{-1}(T(X)), T^{-1}(T(Y))) = d_k(T(X), T(Y))$  for a linear operator  $T : E \rightarrow F$  and symmetric and convex subsets  $X, Y$  of  $E$ .

**Definition 2.7.** We define

$$l_k(T) := \sup\{d_k(T|_N) \mid N \leq E, \dim N \leq k\}.$$

Then  $(l_k)$  has the properties (S2) to (S5) but it may fail to be monotone. Nevertheless, the following estimate holds:

**Theorem 2.8.**

$$l_k(T) \leq d_k(T) \quad \text{and} \quad h_k(T) \leq d_k(T) \leq k^2 l_k(T).$$

For the proof we need the following

**Lemma 2.9** (Bauhardt). Let  $T \in \mathcal{L}(E, F)$ .

(i) For  $x_1, \dots, x_k \in U_E$  and  $L_i := \langle T(x_j) : j \neq i \rangle$  ( $i = 1, \dots, k$ ) the following estimate holds:

$$h_k(T) \geq \frac{1}{k} \min\{\|Q_{L_j}^F T(x_j)\| \mid j = 1, \dots, k\}$$

(ii) For every  $0 < \varepsilon < 1$  and every  $k \in \mathbb{N}$  there exist  $x_1, \dots, x_k \in U_E$  such that

$$\min\{\|Q_{L_j}^F T(x_j)\| \mid j = 1, \dots, k\} \geq \frac{1}{k} d_k(T) (1 - \varepsilon)^2.$$

*Proof.* These are Lemma 1 and Lemma 3 in [6]. □

*Proof of 2.8.* The first inequality of 2.8 follows from (S 3), since  $T|_N = T \circ \iota_N$  where  $\iota_N : N \rightarrow E$  is the embedding.

The inequality  $h_k(T) \leq d_k(T)$  is also a consequence of (S 3).

In order to prove the last inequality, we fix  $0 < \varepsilon < 1$  and  $k \in \mathbb{N}$ . According to 2.9 (ii), there exist  $x_1, \dots, x_k \in U_E$  such that

$$\min\{\|Q_{L_j}^F T(x_j)\| \mid j = 1, \dots, k\} \geq \frac{(1 - \varepsilon)^2}{k} d_k(T)$$

and 2.9 (i) implies

$$h_k(T|_{\langle x_1, \dots, x_k \rangle}) \geq \frac{1}{k} \min\{\|Q_{L_j}^{\langle T(x_1), \dots, T(x_k) \rangle} T(x_j)\| \mid j = 1, \dots, k\}.$$

Taking into consideration  $\|Q_{L_j}^{\langle T(x_1), \dots, T(x_k) \rangle} T(x_j)\| = \|Q_{L_j}^F T(x_j)\|$  and recalling that  $d_k \geq h_k$ , we obtain:

$$d_k(T|_{\langle x_1, \dots, x_k \rangle}) \geq h_k(T|_{\langle x_1, \dots, x_k \rangle}) \geq \frac{(1 - \varepsilon)^2}{k^2} d_k(T).$$

Since  $\varepsilon$  was arbitrary, the assertion follows.

**Corollary 2.10.** *Let  $X$  and  $Y$  be symmetric and convex subsets of a vector space  $E$  with  $d_1(X, Y) < \infty$ . Then*

$$l_k(X, Y) := \sup\{d_k(X \cap N, Y) \mid N \leq E, \dim N \leq k\}$$

*satisfies*

$$l_k(X, Y) \leq d_k(X, Y) \leq k^2 l_k(X, Y).$$

*Proof.* This follows immediately from 2.8 and 2.5. □

### 3. A CHARACTERIZATION OF NON-NUCLEAR METRIZABLE LOCALLY CONVEX SPACES

In this section we define locally convex nuclear vector spaces and prove a characterization of non-nuclear metrizable locally convex spaces.

**Definition 3.1.** A (real or complex) locally convex vector space  $E$  is called a **locally convex nuclear vector space** if for every symmetric and convex neighborhood  $U$  of 0 there exists a symmetric and convex neighborhood  $W$  of 0 such that  $d_k(W, U) \leq 1/k$  for all  $k \in \mathbb{N}$ .

**Remark 3.2.** The statement

$$d_k(W, U) \leq 1/k$$

in the definition of a locally convex nuclear vector can be replaced by

$$d_k(W, U) \leq k^{-\varepsilon}$$

for any  $\varepsilon > 0$ . [This is an easy consequence of 2.6 (i).]

**Proposition 3.3.** *Let  $E$  be a locally convex vector space. The following assertions are equivalent:*

- (i)  *$E$  is a locally convex nuclear vector space,*
- (ii) *for every symmetric and convex 0-neighborhood  $U$  there is a symmetric and convex 0-neighborhood  $W$  such that  $(kd_k(W, U))_{k \in \mathbb{N}} \in \ell^\infty$ ,*
- (iii) *for every  $m \in \mathbb{N}$  and for every symmetric and convex 0-neighborhood  $U$  there is a symmetric and convex 0-neighborhood  $W$  such that  $(k^m d_k(W, U))_{k \in \mathbb{N}} \in \ell^\infty$ .*

*Proof.* This is a direct consequence of 3.2 and 2.6 (iii). □

**Theorem 3.4.** *Let  $E$  be a metrizable locally convex vector space which is not nuclear. Then there exists a symmetric and convex, totally bounded subset  $K$  in  $E$  and a symmetric and convex neighborhood  $U$  of 0 such that  $(k^3 d_k(K, U))$  is unbounded.*

*Conversely, if  $E$  is a locally convex nuclear vector space,  $K \subseteq E$  is a totally bounded symmetric and convex subset,  $U$  is an arbitrary neighborhood of 0, and  $n \in \mathbb{N}$ , then the sequence  $(k^n d_k(K, U))$  is bounded.*

*Proof.* Let us assume first that  $E$  is a metrizable locally convex nuclear vector space which is not nuclear. According to 3.3, there exists a symmetric and convex neighborhood  $U$  of 0 such that for any other symmetric and convex neighborhood  $W$  of 0

$$(kd_k(W, U)) \notin \ell^\infty \quad (*).$$

Let  $(U_n)$  be a decreasing neighborhood basis of 0 consisting of symmetric and convex sets. We may assume that  $U_n \subseteq U$  for all  $n \in \mathbb{N}$ . (\*) implies that for every  $n \in \mathbb{N}$  there exists  $k_n \in \mathbb{N}$  such that  $d_{k_n}(U_n, U) > \frac{n}{k_n}$ .

According to 2.10,  $d_{k_n}(U_n, U) \geq \frac{1}{k_n^2} d_{k_n}(U_n, U) > \frac{n}{k_n^3}$  and hence the following holds:

$$\forall n \in \mathbb{N} \exists N_n \leq E, \dim(N_n) \leq k_n : d_{k_n}(U_n \cap N_n, U) \geq \frac{n}{k_n^3}.$$

The maximal subspace contained in the symmetric and convex set  $U_n \cap N_n$  is  $\{0\}$ . Indeed, suppose this subspace  $L$  is not trivial; consider the canonical projection  $Q_L^E : E \rightarrow E/L$ . Since  $d_1(U_n \cap N_n, U) \leq d_1(U_n, U) < \infty$ , 2.6 implies  $d_{k_n}(U_n \cap N_n, U) = d_{k_n}(Q_L^E(U_n \cap N_n), Q_L^E(U)) = 0$ . The last equality is a consequence of (S 4), since  $\dim Q_F^E(N_n) < k_n$ .

For  $K := \text{conv} \bigcup_{n \in \mathbb{N}} (U_n \cap N_n)$  we obtain

$$d_{k_n}(K, U) \geq d_{k_n}(U_n \cap N_n, U) \geq \frac{n}{k_n^3} \quad \forall n \in \mathbb{N}.$$

Hence it suffices to show that  $K$  is totally bounded.

The sets  $U_n \cap N_n$  are bounded (since they contain only the trivial subspace) and therefore totally bounded, by the equivalence of norms on finite-dimensional vector spaces. For a fixed neighborhood  $W$  of 0 there exists  $m \in \mathbb{N}$  such that  $U_m + U_m \subseteq W$ . Since  $\text{conv} \bigcup_{n=1}^{m-1} U_n \cap N_n$  is totally bounded (cf. (5.1), p.25 and (4.3), p.50 in [17]), there exists a finite set  $F$  such that  $\text{conv} \bigcup_{n=1}^{m-1} U_n \cap N_n \subseteq F + U_m$ . Since the  $(U_n)$  are decreasing and  $U_m$  is convex,  $\text{conv} \bigcup_{n \geq m} U_n \cap N_n \subseteq U_m$ . Hence we obtain:

$$\text{conv} \bigcup_{n \geq 1} U_n \cap N_n \subseteq \text{conv} \bigcup_{n=1}^{m-1} U_n \cap N_n + \text{conv} \bigcup_{n \geq m} U_n \cap N_n \subseteq F + U_m + U_m \subseteq F + W,$$

which shows that  $K$  is totally bounded.

Conversely, assume that  $E$  is a locally convex nuclear vector space,  $K$  is a symmetric and convex totally bounded subset and  $U$  is a symmetric and convex neighborhood of 0. For  $n \in \mathbb{N}$ , there exists a neighborhood  $W$  such that  $d_k(W, U) \leq k^{-n}$  (3.2). Since  $K$  is totally bounded, there exists a finite, symmetric set  $F$  such that  $K \subseteq F + W$ . Let  $f := \dim \langle F \rangle$ . Then  $d_{k+f}(K, U) \leq d_{k+f}(\text{conv} F + W, U) \leq d_k(W, U) \leq k^{-n} \leq c \cdot (k+f)^{-n}$  for suitable  $c > 0$ . The assertion follows.  $\square$

**Corollary 3.5.** *For a metrizable locally convex vector space  $E$  the following assertions are equivalent:*

- (i)  $E$  is nuclear.
- (ii) For every symmetric and convex totally bounded subset  $K$ , for every symmetric and convex 0-neighborhood  $U$  the sequence  $(k^3 d_k(K, U))$  is bounded.

**Remark 3.6.** The vector space  $\mathbb{R}^{(I)}$  endowed with the asterisk-topology (which makes  $\mathbb{R}^{(I)}$  the locally convex direct sum) is not nuclear if  $I$  is uncountable. However, the totally bounded subsets are contained in finite-dimensional subspaces. This implies that for every totally bounded, symmetric and convex set  $K$  and every symmetric and convex 0-neighborhood  $U$  the sequence  $(d_k(K, U))$  is eventually zero, in particular  $(k^n d_k(K, U))$  is bounded for every  $n \in \mathbb{N}$ .

## 4. A PROPERTY OF METRIZABLE NUCLEAR VECTOR GROUPS WHICH ARE NOT NUCLEAR

In this section we recall the definition of locally convex vector groups and nuclear vector groups, which are, roughly speaking, locally convex spaces, respectively locally convex nuclear vector spaces over the scalar field  $\mathbb{R}$  endowed with the discrete topology. These are of interest, since the class of Hausdorff quotients of subgroups of nuclear vector groups coincides with the class of all nuclear groups (to be defined in 5.3). We show a property of non-nuclear metrizable locally convex vector groups which will be applied to prove an analogous property for non-nuclear metrizable groups (5.14). This will be the key lemma in order to show that the character group of a nuclear  $k_\omega$ -group is again nuclear.

**Definition 4.1.** A vector space  $E$  endowed with a Hausdorff group topology  $\mathcal{O}$  is called a **locally convex vector group** if there is a neighborhood basis of 0 consisting of symmetric and convex sets.

A locally convex vector group  $E$  is called a **nuclear vector group** if for every symmetric and convex neighborhood  $U$  of 0 there exists a symmetric and convex neighborhood  $W$  of 0 such that  $d_k(W, U) \leq 1/k$  for all  $k \in \mathbb{N}$ .

**Remark 4.2.** (i) According to Proposition 3 in [11],  $(\mathbb{R}^n, \mathcal{O})$  is a locally convex vector group if and only if there exists a basis  $(b_1, \dots, b_n)$  of  $\mathbb{R}^n$  and  $0 \leq m \leq n$  such that  $\mathcal{O}$  induces the usual topology on  $\langle b_1, \dots, b_m \rangle_{\mathbb{R}}$  and the discrete topology on  $\langle b_{m+1}, \dots, b_n \rangle_{\mathbb{R}}$ .

This shows, that in a locally convex vector group  $(E, \mathcal{O})$  a set of the form  $\{tx \mid |t| \leq 1\}$  ( $x \in E$ ) is not necessarily compact.

- (ii) However, the following holds: Let  $E$  be a locally convex vector group. For every  $\lambda \in \mathbb{R}$ , the scalar multiplication  $m_\lambda : E \rightarrow E$ ,  $x \mapsto \lambda x$  is continuous.
- (iii) This property shows that analogous formulations of 3.2 and 3.3 hold true for nuclear vector groups instead of nuclear vector spaces.

**Lemma 4.3.** *Let  $Y$  be a bounded, symmetric and convex subset of  $\mathbb{R}^n$ . For every  $\varepsilon > 0$ , there exists a finite subset  $F \subseteq Y$  such that  $(1 - \varepsilon)Y \subseteq \text{conv}(F)$ .*

*Proof.* Without loss of generality we may assume that  $\langle Y \rangle = \mathbb{R}^n$  and that  $Y$  is compact if  $\mathbb{R}^n$  is endowed with the usual topology.

[The first assumption is trivial, the second follows from  $Y \subseteq \overline{Y} \subseteq (1 + \varepsilon)Y$ . So if there exists a finite set  $F \subseteq \overline{Y}$  such that  $(1 - \varepsilon)\overline{Y} \subseteq \text{conv}(F)$ , it follows that  $\frac{1 - \varepsilon}{1 + \varepsilon}\overline{Y} \subseteq (1 + \varepsilon)^{-1}\text{conv}F = \text{conv}((1 + \varepsilon)^{-1}F)$ . Since  $1/(1 + \varepsilon)F$  is a finite subset of  $Y$ , it follows that  $Y$  may be assumed to be compact.]

Let  $\mathcal{F}$  be the set of all finite, symmetric subsets of  $Y$  which contain  $n$  linearly independent vectors. This set is not empty. For  $F \in \mathcal{F}$ , the set  $\text{conv}(F)$  has non-empty interior and it is easy to check that  $(1 - \varepsilon)Y \subseteq \bigcup_{F \in \mathcal{F}} \text{int}(\text{conv}(F))$ . Since  $(1 - \varepsilon)Y$  is compact, there exists a finite subcover and the union of the finite sets appearing in the subcover has the desired properties.  $\square$

**Theorem 4.4.** *Let  $E$  be a metrizable, non-nuclear locally convex vector group.*

*There exists a symmetric and convex neighborhood  $U$  of 0 and a null sequence  $(y_n)$  in  $E$  such that for all  $n_0 \in \mathbb{N}$*

$$(k^3 d_k(\text{conv}(\{y_n \mid n \geq n_0\}), U)) \notin \ell^\infty.$$

*Proof.* We fix a decreasing neighborhood basis  $(U_n)$  consisting of symmetric and convex sets. Since  $E$  is not nuclear, there exists, according to 3.3 and 4.2 (iii), a symmetric and convex 0-neighborhood  $U$  such that

$$(kd_k(U_n, U)) \notin \ell^\infty \quad \forall n \in \mathbb{N}.$$

We may assume that  $U_n \subseteq U$  for all  $n \in \mathbb{N}$ .

Hence we have:

$$(4.1) \quad \forall n \in \mathbb{N} \quad \exists k_n \in \mathbb{N} \quad k_n d_{k_n}(U_n, U) > n.$$

According to 2.10,  $l_{k_n}(U_n, U) \geq \frac{1}{k_n^2} d_{k_n}(U_n, U) > \frac{n}{k_n^3}$ , so there exists a sequence  $(E_n)$  of subspaces where  $\dim E_n \leq k_n$  such that

$$(4.2) \quad d_{k_n}(U_n \cap E_n, U) \geq \frac{n}{k_n^3} \quad \forall n \in \mathbb{N}.$$



As in the proof of 3.4 one shows that  $U_n \cap E_n$  is bounded in the finite dimensional space  $E_n$ .

According to 4.3, there exists a finite, symmetric set  $F_n \subseteq U_n \cap E_n$  such that  $\text{conv} F_n \supseteq \frac{1}{2}(U_n \cap E_n)$ ; hence we obtain

$$(4.3) \quad d_{k_n}(\text{conv}(F_n), U) \geq \frac{1}{2} d_{k_n}(U_n \cap E_n, U) \stackrel{(2)}{\geq} \frac{n}{2k_n^3} \quad \forall n \in \mathbb{N}.$$

Since  $F_m \subseteq U_m$ , there exists a null sequence  $(y_j)_{j \in \mathbb{N}}$  such that  $\bigcup_{m \in \mathbb{N}} F_m = \{\pm y_j \mid j \in \mathbb{N}\}$  and which satisfies that for every  $n \in \mathbb{N}$  there exists  $m_n \in \mathbb{N}$  such that  $\bigcup_{m \geq m_n} F_m \subseteq \{\pm y_j \mid j \geq n\}$ . Since

$$d_{k_m}(\text{conv}\{\pm y_j \mid j \geq n\}, U) \geq d_{k_m}(\text{conv}(F_m), U) \stackrel{(3)}{\geq} \frac{m}{2k_m^3} \quad \forall m \geq m_n,$$

we obtain  $(k^3 d_k(\text{conv}\{y_j \mid j \geq n\}, U)) \notin \ell^\infty$  for every  $n \in \mathbb{N}$ .  $\square$

## 5. A PROPERTY OF NON-NUCLEAR METRIZABLE GROUPS

**Notation 5.1.** For an abelian group  $G$  and subsets  $A$  and  $B$  of  $G$  we put

$$(d_k(A, B)) \leq (c_k)$$

(for  $(c_k) \in [0, \infty]^\mathbb{N}$ ) if there exists a vector space  $E$  and symmetric and convex subsets  $X$  and  $Y$  with  $d_k(X, Y) \leq c_k$  (for all  $k \in \mathbb{N}$ ), a subgroup  $H$  of  $E$ , and a homomorphism  $\varphi : H \rightarrow G$  such that  $A \subseteq \varphi(X \cap H)$  and  $\varphi(Y \cap H) \subseteq Y$ .

**Remark 5.2.** Without loss of generality one may assume that the homomorphism  $\varphi$  in 5.1 is surjective. [We may replace  $E$  by  $E \times \mathbb{R}^{(G)}$ ,  $H$  by  $\mathbb{Z}^{(G)}$ ,  $\varphi$  by the surjective homomorphism  $H \times \mathbb{Z}^{(G)} \rightarrow G$ ,  $(h, (k_g)) \mapsto h + \sum k_g g$ , and  $X, Y$  by  $X \times \{0\}, Y \times \{0\}$ , respectively.]

**Definition 5.3.** An abelian Hausdorff group  $G$  is called a **nuclear group** if for every  $m \in \mathbb{N}$ , every  $c > 0$ , and every neighborhood  $U$  of the neutral element 0 there exists another neighborhood  $W$  of 0 such that

$$(d_k(W, U)) \leq (ck^{-m}).$$

**Definition 5.4.** For a subset  $A$  of a topological group  $G$ , the set  $A^\triangleright := \{\chi \in G^\wedge \mid \chi(A) \subseteq \mathbb{T}_+\}$  (with  $\mathbb{T}_+ := \{z \in \mathbb{T} \mid \text{Re } z \geq 0\}$ ) is called **polar** of  $A$ . Conversely, for a subset  $B \subseteq G^\wedge$ , we define  $B^\triangleleft := \{x \in G : \chi(x) \in \mathbb{T}_+ \ \forall \chi \in B\}$ .

A subset  $A$  of a topological group  $G$  is called **quasi-convex** if  $A = (A^\triangleright)^\triangleleft$ . A topological group  $G$  is called **locally quasi-convex** if there is a neighborhood basis of the neutral element consisting of quasi-convex sets.

**Remarks 5.5.** The notation of quasi-convexity is a generalization of the description of convex sets given by the Hahn-Banach theorem.

For an arbitrary subset  $A \subseteq G$ , the set  $(A^\triangleright)^\triangleleft =: \text{qc}(A)$  is the smallest quasi-convex subset which contains  $A$ . It is called the **quasi-convex hull** of  $A$ .

Every locally quasi-convex Hausdorff group is maximally almost periodic (i.e. the continuous characters separate the points).

**Theorem 5.6.** Every nuclear group is locally quasi-convex.

*Proof.* This is Theorem (8.5) in [4].  $\square$

Now we repeat and generalize a representation of locally quasi-convex (nuclear) groups as quotients of subgroups of locally convex (nuclear) vector groups given in (9.6) in [4], since we need a quantitative version here.

Let  $G$  be a locally quasi-convex group. For a nonempty subset  $A \subseteq G$  (not necessarily a neighborhood), we put

$$X_A := \{(x_\chi) \in \mathbb{R}^{G^\wedge} \mid \exists g \in A \text{ such that } e^{2\pi i x_\chi} = \chi(g) \text{ and } |x_\chi| \leq 1/4 \ \forall \chi \in A^\triangleright\},$$

$$Y_A := \text{conv}(X_A),$$

and

$$H_0 := \{(x_\chi)_{\chi \in G^\wedge} \in \mathbb{R}^{G^\wedge} \mid e^{2\pi i x_\chi} = \chi(g) \text{ for some } g \in G \text{ and all } \chi \in G^\wedge\}.$$

Since the characters separate the points, the element  $g \in G$  is unique and hence

$$\varphi_0 : H_0 \rightarrow G, (x_\chi) \mapsto g$$

(if  $\chi(g) = e^{2\pi i x_\chi}$  for all  $\chi \in G^\wedge$ ) is well defined. It is easy to prove that  $\varphi_0$  is a group homomorphism.

For quasi-convex 0-neighborhoods  $U$  and  $W$  such that  $U + U \subseteq W$  it is proved in (9.6) in [4] that  $X_U + X_U \subseteq X_W$  and hence

$$X_U \cap H_0 + X_U \cap H_0 \subseteq X_W \cap H_0, \\ \text{conv}(X_U \cap H_0) + \text{conv}(X_U \cap H_0) \subseteq \text{conv}(X_W \cap H_0), \text{ and } Y_U + Y_U \subseteq Y_W.$$

This shows that  $(Y_U)$  and  $(\text{conv}(X_U \cap H_0))$  (where  $U$  runs through all quasi-convex neighborhoods of 0 in  $G$ ) form neighborhood bases of locally convex vector group topologies  $\mathcal{O}$  and  $\mathcal{O}_{co}$  on  $\mathbb{R}^{G^\wedge}$ . (It is not difficult to verify that they are Hausdorff spaces.) Instead of  $(\mathbb{R}^{G^\wedge}, \mathcal{O}_{co})$  we simply write  $V_0$ .

For a quasi-convex set  $A$  we have

$$(5.1) \quad Y_A \cap H_0 = X_A \cap H_0.$$

The inclusion " $\supseteq$ " is trivial. Conversely, fix  $(x_\chi) \in Y_A \cap H_0$ . Since  $(x_\chi) \in H_0$ , there exists  $g \in G$  such that  $e^{2\pi i x_\chi} = \chi(g)$  for all  $\chi \in G^\wedge$ . Since  $(x_\chi)$  belongs to  $Y_A$   $|x_\chi| \leq 1/4$  for all  $\chi \in A^\flat$  and hence  $\chi(g) \in \mathbb{T}_+$  for all  $\chi \in A^\flat$ , which implies  $g \in \text{qc}(A) = A$  and hence  $(x_\chi) \in X_A \cap H_0$ .

In Lemma 5.12 we will show that  $G$  is a nuclear group if and only if  $V_0$  is a nuclear vector group, both endowed with the topology induced by the neighborhood basis  $(Y_U)$  and  $(\text{conv}(H_0 \cap X_U))$ . For its proof, we will need some lemmas.

Let us first recall that a seminorm  $p$  on a vector space  $E$  is called **pre-Hilbert seminorm** if the parallelogram law  $p(x+y)^2 + p(x-y)^2 = 2(p(x)^2 + p(y)^2)$  holds for all  $x, y \in E$ .

**Lemma 5.7.** *For every  $m \geq 2$  there exists a constant  $c_m > 0$  such that for all symmetric and convex sets  $X$  and  $Y$  in a vector space which satisfy  $d_k(X, Y) \leq ck^{-m}$  for some  $m \geq 2$ , there exist pre-Hilbert seminorms  $p, q$  on  $\langle X \rangle$  such that  $X \subseteq B_p$ ,  $B_q \subseteq Y$  and  $d_k(B_p, B_q) \leq cc_m k^{-m+2}$ .*

*Proof.* This is Lemma (2.14) in [4]. A proof can be found in [2], (18.32) and (18.33).  $\square$

**Lemma 5.8.** *Let  $p$  and  $q$  be pre-Hilbert seminorms on a vector space  $E$  such that  $d_k(B_p, B_q) \rightarrow 0$  and  $d_1(B_p, B_q) < \infty$ . For decreasing sequences of positive real numbers  $(a_k)$  and  $(b_k)$  with  $d_k(B_p, B_q) \leq a_k b_k$  there exists a pre-Hilbert seminorm  $r$  on  $E$  such that  $d_k(B_p, B_r) \leq a_k$  and  $d_k(B_r, B_q) \leq b_k$  for all  $k \in \mathbb{N}$ .*

*Proof.* This is Lemma (2.15) in [4]. A proof can be found in (18.28) in [2].  $\square$

**Lemma 5.9.** *Let  $r$  and  $q$  be pre-Hilbert seminorms on a vector space  $E$  satisfying  $\sum_{k \in \mathbb{N}} d_k(B_r, B_q) < \infty$  and let  $\chi : H \rightarrow \mathbb{T}$  be a homomorphism where  $H$  is a subgroup of  $E$ . If  $\chi(B_q \cap H) \subseteq \mathbb{T}_+$  then there exists a linear mapping  $f : E \rightarrow \mathbb{R}$  such that  $e^{2\pi i f}|_H = \chi$  and*

$$\sup\{|f(x)| \mid x \in B_r\} \leq \frac{21}{2\pi} \sum_{k \in \mathbb{N}} d_k(B_r, B_q).$$

*Proof.* This is (8.1) in [4]. (The proof of the sharper estimate can be found in (19.14) (ii) in [2].)  $\square$

**Lemma 5.10.** *Let  $p$  and  $q$  be pre-Hilbert seminorms on a vector space  $E$  with  $\sum_{k \in \mathbb{N}} d_k(B_p, B_q)^2 < \frac{1}{4}$ . Then for any subgroup  $H$  of  $E$  the following estimate holds:*

$$d_k(\text{conv}(H \cap B_p), \text{conv}(H \cap B_q)) \leq 2d_k(B_p, B_q).$$

*Proof.* This is Corollary (3.20) in [4].  $\square$

**Lemma 5.11.** *Let  $A$  be an arbitrary nonempty subset and let  $B \supseteq A$  be a quasi-convex subset of an abelian topological group  $G$  such that  $(d_k(A, B)) \leq (ck^{-m})$ . If  $(8\pi cc_m)^2 \sum_{k \in \mathbb{N}} (k^{-m+4})^2 < 1/4$ , then*

$$d_k(Y_A, Y_B) \leq 16\pi cc_m k^{-m+4}.$$

*If  $(16\pi cc_m c_{m-4})^2 \sum_{k=1}^{\infty} (k^{-m+6})^2 < 1/4$  then*

$$d_k(\text{conv}(X_A \cap H_0), \text{conv}(X_B \cap H_0)) \leq 32\pi cc_m c_{m-4} k^{-m+6}.$$

*Proof.* Observe that  $m \geq 5$ .

By assumption, there is a vector space  $E$ , a subgroup  $H$  of  $E$  and a homomorphism  $\varphi : H \rightarrow G$  and there are symmetric and convex sets  $X$  and  $Y$  such that

$$(5.2) \quad d_k(X, Y) \leq ck^{-m}, \quad A \subseteq \varphi(X \cap H), \quad \text{and} \quad \varphi(Y \cap H) \subseteq B.$$

According to 5.7, there are pre-Hilbert seminorms  $p$  and  $q$  defined on  $\langle X \rangle$  such that

$$(5.3) \quad d_k(B_p, B_q) \leq cc_m k^{-m+2} \quad \text{and} \quad X \subseteq B_p, \quad B_q \subseteq Y.$$



As a consequence of 5.8, there exists a pre-Hilbert seminorm  $r$  such that

$$(5.4) \quad d_k(B_p, B_r) \leq 8\pi c c_m k^{-m+4} \quad \text{and} \quad d_k(B_r, B_q) \leq \frac{1}{8\pi} k^{-2}.$$

Let us fix  $\chi \in B^\flat$ . Then  $\chi \circ \varphi(B_q \cap H) \stackrel{(6)}{\subseteq} \chi(\varphi(Y \cap H)) \stackrel{(5)}{\subseteq} \chi(B) \subseteq \mathbb{T}_+$ . According to 5.9, there is a linear function  $f_\chi : \langle X \rangle \rightarrow \mathbb{R}$  such that

$$e^{2\pi i f_\chi}|_H = \chi \circ \varphi \quad \text{and}$$

$$(5.5) \quad \sup\{|f_\chi(x)| \mid x \in B_r\} \leq \frac{21}{2\pi} \sum_{k \in \mathbb{N}} d_k(B_r, B_q) = \frac{21}{2\pi} \cdot \frac{1}{8\pi} \cdot \frac{\pi^2}{6} < \frac{1}{4}.$$

For  $\chi \notin B^\flat$  we put  $f_\chi \equiv 0$ . We define

$$\Phi : \langle X \rangle \rightarrow V_0 = \mathbb{R}^{G^\wedge}, \quad x \mapsto (f_\chi(x))_{\chi \in G^\wedge}$$

and let

$$P : \mathbb{R}^{G^\wedge} \rightarrow \mathbb{R}^{B^\flat} \times \{0\}^{G^\wedge \setminus B^\flat}$$

be the canonical projection.

Then the following inclusions hold:

$$(5.6) \quad P(H_0) \cap \Phi(B_r) \subseteq P(X_B \cap H_0)$$

$$(5.7) \quad P(X_A) \subseteq P(H_0) \cap \Phi(B_p)$$

In order to prove (9), we fix  $(x_\chi) \in P(H_0) \cap \Phi(B_r)$ . Since  $(x_\chi) \in \Phi(B_r)$ , there exists  $x \in B_r$  such that  $x_\chi = f_\chi(x)$  for all  $\chi \in B^\flat$ . In particular,  $e^{2\pi i x_\chi} = e^{2\pi i f_\chi(x)}$  for all  $\chi \in B^\flat$ . On the other hand,  $(x_\chi) \in P(H_0)$ , which means that there exists  $g \in G$  such that  $\chi(g) = e^{2\pi i x_\chi} = e^{2\pi i f_\chi(x)}$  for all  $\chi \in B^\flat$ . Since  $x \in B_r$ ,  $|f_\chi(x)| \leq 1/4$  (by (8)). This shows that  $\chi(g) \in \mathbb{T}_+$  for all  $\chi \in B^\flat$  which implies  $g \in B$ , since  $B$  was assumed to be quasi-convex. It follows that  $(x_\chi) \in P(X_B \cap H_0)$ .

Now we are going to prove (10). Observe first that  $A \subseteq B$  implies  $B^\flat \subseteq A^\flat$ . We fix  $(y_\chi) \in P(X_A)$ . This means that  $y_\chi = 0$  for all  $\chi \notin B^\flat$  and  $|y_\chi| \leq 1/4$  for all  $\chi \in B^\flat \subseteq A^\flat$ ; further, there exists  $g \in A$  such that  $\chi(g) = e^{2\pi i y_\chi}$  ( $\forall \chi \in B^\flat$ ). The latter equality implies immediately  $P((y_\chi)) \in P(H_0)$ .

Since  $A \subseteq \varphi(H \cap B_p)$  (according to (5) and (6)), there exists  $x \in H \cap B_p$  with  $g = \varphi(x)$ . We want to show that  $(y_\chi) = \Phi(x) = (f_\chi(x))$ . Therefore we fix  $\chi \in B^\flat$ . By assumption and (7),  $\sum d_k(B_p, B_r)^2 < 1/4$ , in particular  $d_1(B_p, B_r) < 1$ , which implies  $B_p \subseteq B_r$ . As a consequence of this,  $|f_\chi(x)| \leq 1/4$  for all  $\chi \in B^\flat$  (by (8)). The assertion follows from  $e^{2\pi i f_\chi(x)} = \chi(\varphi(x)) = \chi(g) = e^{2\pi i y_\chi}$  and the fact that  $|y_\chi| \leq 1/4$  and  $|f_\chi(x)| \leq 1/4$ . For  $\chi \notin B^\flat$ ,  $y_\chi = f_\chi(x) = 0$ .

$$(5.8) \quad Y_A \subseteq P^{-1}(P(Y_A)) \quad \text{and} \quad Y_B = P^{-1}(P(Y_B))$$

The inclusion is trivial while the equality is a consequence of the fact that  $Y_B + \ker P = Y_B$ .

We obtain:

$$(5.9) \quad \begin{aligned} d_k(Y_A, Y_B) &\stackrel{(11)}{\leq} d_k(P^{-1}(P(Y_A)), P^{-1}(P(Y_B))) \\ &\stackrel{2.6(v)}{=} d_k(P(Y_A), P(Y_B)) \\ &= d_k(\text{conv}(P(X_A)), \text{conv}(P(X_B))) \\ &\stackrel{(9),(10)}{\leq} d_k(\text{conv}(P(H_0) \cap \Phi(B_p)), \text{conv}(P(H_0) \cap \Phi(B_r))) \end{aligned}$$

By assumption and (7),  $\sum_{k \in \mathbb{N}} d_k(B_p, B_r)^2 < \frac{1}{4}$ , hence 5.10 implies

$$(5.10) \quad \begin{aligned} d_k(\text{conv}(P(H_0) \cap \Phi(B_p)), \text{conv}(P(H_0) \cap \Phi(B_r))) &\leq \\ 2d_k(\Phi(B_p), \Phi(B_r)) &\leq 2d_k(B_p, B_r). \end{aligned}$$

(Observe that the Minkowski functionals associated to  $\Phi(B_p)$  and  $\Phi(B_r)$  defined on  $\Phi(\langle X \rangle)$  satisfy the parallelogram law.) Combining (12), (13), and (7), we obtain:

$$(5.11) \quad d_k(Y_A, Y_B) \leq 16\pi c c_m k^{-m+4}.$$

Let us assume now that  $(16\pi c c_m c_{m-4})^2 \sum_{k=1}^{\infty} (k^{-m+6})^2 < 1/4$ . This implies in particular  $m \geq 7$ . Hence there exist, according to 5.7 and (14), pre-Hilbert seminorms  $p'$  and  $q'$  defined on  $\langle Y_A \rangle$  such that

$$Y_A \subseteq B_{p'}, \quad B_{q'} \subseteq Y_B, \quad \text{and} \quad d_k(B_{p'}, B_{q'}) \leq 16\pi c c_m c_{m-4} k^{-m+6}.$$

Recall that  $X_A \cap H_0 \subseteq Y_A \cap H_0$  holds trivially and  $X_B \cap H_0 = Y_B \cap H_0$  was proved in (4). Hence we obtain

$$\begin{aligned} d_k(\text{conv}(X_A \cap H_0), \text{conv}(X_B \cap H_0)) &\leq d_k(\text{conv}(Y_A \cap H_0), \text{conv}(Y_B \cap H_0)) \\ &\leq d_k(\text{conv}(B_{p'} \cap H_0), \text{conv}(B_{q'} \cap H_0)) \stackrel{5.10}{\leq} 2d_k(B_{p'}, B_{q'}) \\ &\leq 32\pi c c_m c_{m-4} k^{-m+6}, \end{aligned}$$

which had to be shown.  $\square$

**Corollary 5.12.** *Let  $G$  be a locally quasi-convex group. The following assertions are equivalent:*

- (i)  $G$  is a nuclear group.
- (ii)  $\mathbb{R}^{G^\wedge}$  with the topology induced by the neighborhood basis  $(Y_U)$  (where  $U$  runs through all quasi-convex neighborhoods of  $e \in G$ ) is a nuclear vector group.
- (iii)  $V_0$  with the topology induced by the neighborhood basis  $(\text{conv}(X_U \cap H_0))$  (where  $U$  runs through all quasi-convex neighborhoods of  $e \in G$ ) is a nuclear vector group.

*Proof.* 5.11 implies (i)  $\Rightarrow$  (ii) and (i)  $\Rightarrow$  (iii).

Conversely, for any quasi-convex neighborhood  $U$  we have

$$\varphi_0(H_0 \cap Y_U) \stackrel{(4)}{=} \varphi_0(H_0 \cap X_U) = U.$$

Since  $X_U \cap H_0 \subseteq \text{conv}(X_U \cap H_0) \cap H_0 \subseteq Y_U \cap H_0 \stackrel{(4)}{=} X_U \cap H_0$ , we obtain that both vector group topologies induce the same topology on  $H_0$  w.r.t. which  $\varphi_0$  is continuous and open. Since the class of nuclear groups is closed w.r.t. forming subgroups and Hausdorff quotients ((7.2) in [4]),  $G$  is nuclear and hence (ii)  $\Rightarrow$  (i) and (iii)  $\Rightarrow$  (i).  $\square$

**Lemma 5.13.** *Let  $(x_n)$  be a null sequence in  $V_0$  contained in  $H_0 \cap X_U$  where  $U$  is a quasi-convex neighborhood of the neutral element of  $G$ . The subspace  $N := \langle \{0\}^{U^\flat} \times \mathbb{Z}^{G^\wedge \setminus U^\flat} \rangle$  is contained in  $\text{conv}(H_0 \cap X_U)$  and*

$$\text{conv}\{\pm x_n \mid n \in \mathbb{N}\} \subseteq N + \text{conv}(H_0 \cap X_A)$$

where  $A := \{\pm \varphi_0(x_n) \mid n \in \mathbb{N}\}$ .

*Proof.* It is obvious, that  $\{0\}^{U^\flat} \times \mathbb{Z}^{G^\wedge \setminus U^\flat}$  is a subset of  $H_0 \cap X_U$ . Hence  $N \subseteq \text{conv}(H_0 \cap X_U)$ .

Without loss of generality we may assume that  $\{x_n \mid n \in \mathbb{N}\} = \{\pm x_n \mid n \in \mathbb{N}\}$ . Fix any  $(x_\chi) = x := x_n \in H_0 \cap X_U$ . This means that there exists  $g \in G$  such that  $|x_\chi| \leq 1/4$  for all  $\chi \in U^\flat$  and  $e^{2\pi i x_\chi} = \chi(g)$  for all  $\chi \in G^\wedge$ . In particular, for  $\chi \in U^\flat$ , we have  $\chi(g) \in \mathbb{T}_+$ , which implies  $g \in U$ .

For those  $\chi \in A^\flat \setminus U^\flat$  with  $|x_\chi| > 1/4$ , we choose  $k_\chi \in \mathbb{Z}$  such that  $|x_\chi - k_\chi| \leq 1/4$ . In all other coordinates we define  $k_\chi := 0$ . Then  $(x_\chi) - (k_\chi) \in H_0 \cap X_A$ , since  $g = \varphi_0(x) = \varphi_0((x_\chi - k_\chi)) \in A$ . The assertion follows, since the right hand side is convex.  $\square$

**Theorem 5.14.** *Let  $G$  be a metrizable, locally quasi-convex Hausdorff group which is not nuclear. Then there is a null sequence  $(g_n)$  in  $G$  and a quasi-convex neighborhood  $U$  of the neutral element such that for no  $n_0 \in \mathbb{N}$  the estimate*

$$(d_k(\{\pm g_n : n \geq n_0\} \cup \{0\}, U)) \leq (ck^{-9})$$

holds if  $c < \frac{\sqrt[4]{945}}{32\pi^4 c_5 c_9}$ .

*Proof.* Let  $(U_n)$  be a decreasing neighborhood basis of the neutral element of  $G$  consisting of quasi-convex sets. Let  $V_0$ ,  $H_0$  and  $\varphi_0$  be as introduced above. Since  $G$  is not a nuclear group,  $V_0$  is not a nuclear vector group either (5.12). According to 4.4, there exists a null sequence  $(y_n)$  in  $V_0$  and a symmetric and convex neighborhood of 0 (we may assume that it is of the form  $\text{conv}(H_0 \cap X_U)$  for some a quasi-convex neighborhood  $U$  of the neutral element in  $G$ ) such that for every  $m \in \mathbb{N}$

$$(k^3 d_k(\text{conv}\{\pm y_n \mid n \geq m\}, \text{conv}(H_0 \cap X_U))) \notin \ell^\infty$$

holds.

We choose  $n_0$  such that  $y_n \in \text{conv}(H_0 \cap X_U)$  for all  $n \geq n_0$ . Since every  $y \in \text{conv}(H_0 \cap X_{U_k})$  belongs to  $\text{conv}(F_k)$  for a suitable finite set  $F_k \subseteq H_0 \cap X_{U_k}$ , there is a null sequence  $(x_k)$  in  $H_0$  such that  $\{\pm y_n \mid n \geq n_0\} \subseteq \text{conv}\{\pm x_k \mid k \geq 1\}$  and such that for every  $m \in \mathbb{N}$  there exists  $n_m \in \mathbb{N}$  with

$$(5.12) \quad \{\pm y_n \mid n \geq n_m\} \subseteq \text{conv}\{\pm x_n \mid n \geq m\}.$$

In particular,

$$(5.13) \quad (k^3 d_k(\text{conv}\{\pm x_n : n \geq m\}, \text{conv}(H_0 \cap U))) \notin \ell^\infty \quad \forall m \in \mathbb{N}.$$

Let  $g_n := \varphi_0(x_n)$  and  $A_m := \{\pm g_n \mid n \geq m\} \cup \{0\}$ . According to 5.13,

$$(5.14) \quad \text{conv}\{\pm x_n \mid n \geq m\} \subseteq N + \text{conv}(H_0 \cap X_{A_m})$$

and  $N \subseteq \text{conv}(X_U \cap H_0)$  for  $N = \langle \{0\}^{G^\wedge \setminus U^\triangleright} \times \mathbb{Z}^{U^\triangleright} \rangle$ . Hence we obtain:

$$(5.15) \quad \begin{aligned} d_k(\text{conv}(X_{A_m} \cap H_0), \text{conv}(X_U \cap H_0)) &\stackrel{2.6(iii)}{=} \\ d_k(\text{conv}(X_{A_m} \cap H_0) + N, \text{conv}(X_U \cap H_0)) &\stackrel{(17)}{\geq} \\ d_k(\text{conv}(\{\pm x_n \mid n \geq m\}), \text{conv}(X_U \cap H_0)) &\stackrel{(15)}{\geq} \\ d_k(\text{conv}(\{\pm y_n \mid n \geq n_m\}), \text{conv}(X_U \cap H_0)). \end{aligned}$$

Assume now that for some  $m \in \mathbb{N}$

$$(d_k(A_m, U)) \leq (ck^{-9}) \quad \text{where } c < \frac{\sqrt[4]{945}}{32\pi^4 c_5 c_9}.$$

5.11 implies that

$$(5.16) \quad d_k(\text{conv}(H_0 \cap X_{A_m}), \text{conv}(H_0 \cap X_U)) \leq 32\pi c c_5 c_9 k^{-3},$$

$$\text{since } (16\pi c c_5 c_9)^2 \underbrace{\sum_{k \in \mathbb{N}} k^{-6}}_{\frac{\pi^6}{\sqrt{945}}} < 1/4.$$

Combining (18) and (19), we obtain that

$$(k^3 d_k(\text{conv}\{y_n \mid n \geq n_m\}, \text{conv}(X_U \cap H_0)))$$

is bounded. This contradiction completes the proof.  $\square$

## 6. STRONGLY REFLEXIVE GROUPS

In this section we prove first that the dual space of a locally convex nuclear  $k_\omega$ -space is again nuclear. Afterwards we establish a technical lemma in order to prove the analogue for the group case. Combining these results with well known properties of nuclear groups and  $k_\omega$ -groups we obtain that nuclear  $k_\omega$ -groups are strongly reflexive.

**Definition 6.1.** A Hausdorff topological space  $X$  is called  $k_\omega$ -**space**, if it has a countable cobasis for the compact sets and the topology is the final topology induced by the compact subsets.

A Hausdorff space  $(X, \mathcal{O})$  is called a  $k$ -**space** if  $\mathcal{O}$  coincides with the final topology induced by all compact subsets of  $X$ .

Observe that every  $k_\omega$ -space is a  $k$ -space.

**Proposition 6.2.** *If  $G$  is an abelian metrizable group then its character group is complete and a  $k_\omega$ -space.*

*If  $G$  is a topological group and a  $k_\omega$ -space, then its dual group is complete and metrizable.*

*Proof.* The first assertion is a consequence of (4.7) in [2] or Theorem 1 in [7] and (1.11) in [4].

The second assertion is (2.8) in [2] and (1.11) in [4]. □

Before stating the next proposition, we recall the following

**Notation 6.3.** *For an abelian topological group  $(G, \tau)$  we denote by  $G^\wedge$  the group of all continuous characters  $\chi : G \rightarrow \mathbb{T}$ . Endowed with the compact-open topology,  $G^\wedge$  is an abelian Hausdorff group. This allows to define  $G^{\wedge\wedge}$ . Let*

$$\alpha_G : G \longrightarrow G^{\wedge\wedge}, \quad x \mapsto \alpha_G(x) : \chi \mapsto \chi(x)$$

*denote the canonical homomorphism which in general is neither injective, nor continuous, nor surjective, nor open.*

**Proposition 6.4.** *If  $G$  is a  $k$ -space then  $\alpha_G$  is continuous. (This is equivalent to: every compact subset of  $G^\wedge$  is equicontinuous.)*

*Proof.* This is a consequence of (1.1) and (2.3) in [12] □

**Lemma 6.5.** *Let  $A$  and  $B$  be symmetric and convex subsets of a vector space  $E$ . Assume that  $B$  is absorbing. Then*

$$d_k(B^\triangleright, A^\triangleright) \leq k d_k(A, B)$$

*where the polars are formed in the algebraic dual  $E^*$  of  $E$ .*

*Proof.* Let  $A \subseteq cB + L$  where  $L$  is an at most  $(k-1)$ -dimensional subspace of  $E$ . Then  $B^\triangleright \cap L^\perp \subseteq cA^\triangleright$ . Consider the vector space  $E_B^* := \{\varphi \in E^* \mid \varphi(B) \text{ is bounded}\} \supseteq B^\triangleright$ ; the Minkowski functional  $p_{B^\triangleright}$  is a norm, since  $B$  was assumed to be absorbing. By Auerbach's Lemma, there exists a projection  $\pi : E_B^* \rightarrow L^\perp \cap E_B^*$  with  $\pi(B^\triangleright) \subseteq kB^\triangleright \cap L^\perp$ . The assertion follows from  $B^\triangleright \subseteq \ker \pi + \pi(B^\triangleright) \subseteq \ker \pi + k(B^\triangleright \cap L^\perp)$  and the fact that  $\dim(\ker \pi) = \dim L \leq k-1$ . □

**Theorem 6.6.** *Let  $E$  be a locally convex vector space which is (as a topological space) a  $k_\omega$ -space. Then  $E'_{co}$ , the dual space of  $E$  endowed with the compact-open topology, is a nuclear space if and only if  $E$  is nuclear.*

*Proof.* Since  $E$  is a  $k_\omega$ -space,  $E'_{co}$  is complete and metrizable (6.2 and 1.1). Hence  $E'_{co}$  is a Fréchet space.

Let us suppose first that  $E$  is a nuclear locally convex vector space and let us assume that  $E'_{co}$  is not nuclear. According to 3.4, there exists a symmetric and convex neighborhood  $U$  of 0 in  $E'_{co}$  and a totally bounded convex symmetric subset  $K$  of  $E'_{co}$  such that  $(k^3 d_k(K, U))$  is unbounded. Replacing  $K$  by its closure, we may assume that  $K$  is compact.

According to 6.4,  $K$  is equicontinuous, which means that  $B := K^\triangleleft = \{x \in E : |\varphi(x)| \leq 1 \ \forall \varphi \in K\}$  is a neighborhood of 0 in  $E$ . Since  $E$  was assumed to be nuclear, there exists another symmetric and convex 0-neighborhood  $A$  such that  $d_k(A, B) \leq k^{-4}$  for all  $k \in \mathbb{N}$ . Now 6.5 implies that  $d_k(B^\triangleright, A^\triangleright) \leq k^{-3}$  holds for all  $k \in \mathbb{N}$ .

Observe the  $K \subseteq B^\triangleright$ . Since  $A^\triangleright$  is compact and hence bounded, there exists  $c > 0$  such that  $A^\triangleright \subseteq cU$ . So we obtain:

$$d_k(K, U) \stackrel{2.6(iii)}{=} cd_k(K, cU) \leq cd_k(B^\triangleright, A^\triangleright) \leq ck^{-3},$$

which implies that the sequence  $(k^3 d_k(K, U))_{k \in \mathbb{N}}$  is bounded in contradiction to our assumption.

Conversely, let us assume that  $E'_{co}$  is nuclear. According to 1.2,  $(E'_{co})'_{co}$  is a locally convex nuclear vector space. Since the evaluation mapping  $E \rightarrow (E'_{co})'_{co}$ ,  $x \mapsto (f \mapsto f(x))$  is an embedding,  $E$  is a nuclear space as well. □

In order to prove an analogous result for groups, we need one more technical lemma, which allows us to manipulate the factor  $c$  in the setting  $(d_k(A, B)) \leq (ck^{-m})$ .

**Lemma 6.7.** *Let  $(g_n)$  be a null-sequence in a complete Hausdorff topological group  $G$  and let  $W$  be a closed neighborhood of 0 such that  $(d_k(\{\pm g_n : n \in \mathbb{N}\}, W)) \leq (ck^{-m})$  for some  $m \geq 1$ . For every  $c_1 > 0$  there exists  $n_0 \in \mathbb{N}$  such that*

$$(d_k(\{\pm g_n : n \geq n_0\}, W)) \leq (c_1 k^{-m+1}).$$

*Proof.* By assumption, there exists a vector space  $E$ , a subgroup  $H$  of  $E$  and a homomorphism  $\varphi : H \rightarrow G$  and symmetric and convex subsets  $X, Y$  of  $E$  which satisfy  $d_k(X, Y) \leq ck^{-m}$  for all  $k \in \mathbb{N}$  and

$$(6.1) \quad \{\pm g_n : n \in \mathbb{N}\} \subseteq \varphi(X \cap H) \text{ and } \varphi(Y \cap H) \subseteq W.$$

Without loss of generality, we may assume that  $E = \langle Y \rangle$ . Let  $\tilde{E}$  be the completion of the seminormed space  $(E, p_Y)$  (where  $p_Y$  is the Minkowski functional of  $Y$ ) and observe that (20) implies that  $\varphi : H \rightarrow G$  is continuous w.r.t. the induced topology. According to (10.19) in [10],  $\varphi$  can be extended to a continuous homomorphism  $\bar{\varphi} : \bar{H} \rightarrow G$  (since  $G$  is complete). Hence there exists  $\eta > 0$  such that  $\bar{\varphi}(\bar{H} \cap \eta \bar{Y}) \subseteq W$ . Observe further that  $X \subseteq cY + L$  (for a subspace  $L \leq E$ ) implies

$$\bar{X} \subseteq X + \varepsilon \bar{Y} \subseteq (c + \varepsilon) \bar{Y} + L \quad \forall \varepsilon > 0$$

and hence  $d_k(\bar{X}, \bar{Y}) \leq d_k(X, Y)$ .

Hence we may assume that  $E$  is complete,  $X, Y$ , and  $H$  are closed in  $E$ ; nevertheless, have to pay for this improvement by a (possibly) larger constant, i.e.  $d_k(X, Y) \leq \frac{c}{\eta} k^{-m}$  as replaced  $Y$  by  $\eta \bar{Y}$  ( $X$  by  $\bar{X}$  and  $H$  by  $\bar{H}$ ).

Let  $K := \ker \varphi$ . According to (20), it is possible to fix for every  $n \in \mathbb{N}$  an element  $h_n \in X \cap H$  with  $\varphi(h_n) = g_n$ . Let  $A$  denote the set of accumulation points of the sequence  $(h_n)_{n \in \mathbb{N}}$ . Since  $H \cap X$  is closed, we have  $A \subseteq H \cap X$ . We want to show that  $A \subseteq K \cap X$ . It remains to show that  $A \subseteq K$ . For that we fix  $a \in A$ . There exists a subsequence  $(h_{n_k})$  of  $(h_n)$  converging to  $a$ . Since  $\varphi(h_{n_k}) = g_{n_k} \rightarrow 0$  and (by the continuity of  $\varphi$ )  $\varphi(h_{n_k}) \rightarrow \varphi(a)$ , we obtain  $\varphi(a) = 0$  since  $G$  is a Hausdorff group. This means  $a \in K$ .

We fix  $k_0 \in \mathbb{N}$  such that  $\frac{2c}{\eta k_0} \leq c_1$  and define  $\varepsilon := c_1 k_0^{-m+1}$ .

Since  $(d_k(X, Y))$  is a null-sequence,  $X$  is a totally bounded subset of  $(E, p_Y)$  (cf. (9.1.4) in [14]). In particular, all but a finite number of elements of the sequence  $(h_n)_{n \in \mathbb{N}}$  are contained in  $A + \varepsilon Y$ . Indeed, assume there exist infinitely many members of the sequence  $(h_n)$  outside  $A + \varepsilon Y$ . This infinite set must have an accumulation point  $b$ . Of course,  $b \in A$  and the neighborhood  $b + \varepsilon Y$  contains an infinite number of them in contradiction to our assumption.

We fix  $n_0$  such that  $h_n \in A + \varepsilon Y$  for all  $n \geq n_0$ .

For every  $n \geq n_0$  we define  $h'_n := h_n - a_n$  for suitable  $a_n \in A$  such that  $h'_n \in \varepsilon Y$ . Hence  $\{h'_n \mid n \geq n_0\} \subseteq (X \cap H) - A \subseteq 2X$  and therefore

$$X' := \text{conv}\{\pm h'_n : n \geq n_0\} \subseteq 2X \cap \varepsilon Y.$$

For  $k \geq k_0$  we obtain

$$d_k(X', Y) \stackrel{2.6(iii)}{\leq} 2d_k(X, Y) \leq \frac{2c}{\eta} k^{-m} \leq \frac{2c}{\eta k_0} k^{-m+1} \leq c_1 k^{-m+1}$$

and for  $k < k_0$ :

$$d_k(X', Y) \leq d_1(X', Y) \leq \varepsilon = c_1 k_0^{-m+1} \leq c_1 k^{-m+1}.$$

Now we have:  $\{\pm g_n \mid n \geq n_0\} \subseteq \varphi(X' \cap H)$ ,  $\varphi(Y \cap H) \subseteq W$  and  $d_k(X', Y) \leq c_1 k^{-m+1}$ , which completes the proof.  $\square$

**Proposition 6.8.** *Let  $G$  be a nuclear group. For every totally bounded subset  $S \subseteq G$  and every neighborhood  $U$  of the neutral element and all  $n \in \mathbb{N}$ , there exists a constant  $c > 0$  such that  $(d_k(S, U)) \leq (ck^{-n})$ .*

*Proof.* Since  $G$  is a nuclear group, there exists neighborhood  $W$  of 0 such that  $(d_k(W, U)) \leq (k^{-n})$ . By definition, this means, there exists a vector space  $E$  and symmetric and convex subsets  $X$  and  $Y$  such that  $d_k(X, Y) \leq k^{-n}$  for all  $k \in \mathbb{N}$  and further a subgroup  $H$  of  $E$  and a surjective group homomorphism,  $\varphi : H \rightarrow G$  such that  $W \subseteq \varphi(X \cap H)$  and  $\varphi(Y \cap U) \subseteq U$ .

Since  $S$  is totally bounded and  $\varphi$  is surjective, there exists a finite set  $F \subseteq H$  such that

$$S \subseteq \varphi(F) + W \subseteq \varphi(F + (X \cap H)) \subseteq \varphi((X + F) \cap H).$$

If  $X \subseteq cY + L$  then  $X + \text{conv} F \subseteq cY + L + \langle F \rangle$ , which implies

$$d_{k+f}(X + \text{conv} F, Y) \leq d_k(X, Y)$$

where  $f := \dim\langle F \rangle$ . In particular,

$$(k + f)^n d_{k+f}(X + \text{conv} F, Y) \leq k^n (1 + f)^n d_k(X, Y) \leq (1 + f)^n.$$

Since  $S \subseteq \varphi((F + X) \cap H) \subseteq \varphi((\text{conv} F + X) \cap H)$ , the assertion follows.  $\square$

**Lemma 6.9.** *For subsets  $A$  and  $B$  of an abelian topological group  $G$  for which  $(d_k(A, B)) \leq (ck^{-n})$  holds for some  $n \geq 5$ , the polars satisfy  $(d_k(B^\triangleright, A^\triangleright)) \leq (cc_n k^{-n+5})$ .*

*Proof.* This is (16.4) in [4].  $\square$

**Corollary 6.10.** *Let  $G$  be a nuclear group such that  $\alpha_G$  is continuous.*

*For every compact subset  $K \subseteq G^\wedge$  and every neighborhood  $W$  of the neutral element of  $G^\wedge$  and every  $n \in \mathbb{N}$ , there exists  $c > 0$  such that*

$$(d_k(K, U)) \leq (ck^{-n}).$$

*Proof.* By the definition of the compact-open topology, there exists a compact subset  $S \subseteq G$  such that  $S^\triangleright \subseteq U$ . Since  $\alpha_G$  is continuous,  $K$  is equicontinuous, which means that  $K \subseteq W^\triangleright$  for a suitable neighborhood  $W$  of the neutral element of  $G$ . According to 6.8, we have  $(d_k(S, W)) \leq (ck^{n+5})$  for a suitable constant  $c > 0$ . 6.9 implies  $(d_k(W^\triangleright, S^\triangleright)) \leq (cc_{n+5} k^{-n})$  and hence  $(d_k(K, U)) \leq (d_k(W^\triangleright, S^\triangleright)) \leq (cc_{n+5} k^{-n})$ , which completes the proof.  $\square$

Now we are prepared to prove the main theorem of this article:

**Theorem 6.11.** *Let  $G$  be a nuclear  $k_\omega$ -group. Then  $G^\wedge$  is a completely metrizable nuclear group.*

*Proof.* Since  $G$  is a  $k_\omega$ -group,  $\alpha_G$  is continuous and  $G^\wedge$  is completely metrizable (6.2). Let us assume that  $G^\wedge$  is not nuclear. According to 5.14, there exists a null sequence  $(g_n)$  and a quasi-convex (and hence closed) neighborhood  $U$  such that

$$(d_k(\{\pm g_n \mid n \geq n_0\} \cup \{0\}, U)) \leq (ck^{-9})$$

does not hold for any  $n_0 \in \mathbb{N}$  if  $c < \frac{\sqrt[4]{945}}{32\pi^4 c_5 c_9} =: c_0$ . Since  $\{\pm g_n \mid n \geq n_0\} \cup \{0\}$  is compact and  $U$  is a neighborhood of 0, there exists by 6.10 a constant  $c_1 > 0$  such that

$$(d_k(\{\pm g_n \mid n \geq 0\} \cup \{0\}, U)) \leq (c_1 k^{-10}).$$

6.7 implies the existence of  $n_0 \in \mathbb{N}$  such that

$$(d_k(\{\pm g_n \mid n \geq n_0\} \cup \{0\}, U)) \leq (c_2 k^{-9})$$

where  $c_2 = c_0/2$ . This contradiction completes the proof.  $\square$

**Theorem 6.12.** *Every nuclear  $k_\omega$ -group is strongly reflexive.*

*Proof.* Let  $G$  be a nuclear  $k_\omega$ -group. According to [15], every  $k_\omega$ -group is complete. (21.5) in [2] implies that  $\alpha_G$  is an open isomorphism.  $G$  being a  $k$ -space implies that  $\alpha_G$  is continuous (6.4) and hence  $\alpha_G$  must be a topological isomorphism which means that  $G$  is reflexive.

Every closed subgroup and every Hausdorff quotient group of a  $k_\omega$ -group, resp. nuclear group is a  $k_\omega$ -group, resp. nuclear. In particular, all closed subgroups and Hausdorff quotient groups of  $G$  are reflexive.

According to 6.11,  $G^\wedge$  is a completely metrizable nuclear group and hence, due to (17.3) in [4],  $G^\wedge$  is strongly reflexive. Therefore, all closed subgroups and Hausdorff quotient groups of  $G^\wedge$  are reflexive as well.  $\square$

**Definition 6.13.** An abelian topological group  $G$  is called **almost metrizable** if there is a compact subgroup  $H$  such that  $G/H$  is metrizable.

**Examples 6.14.** Every metrizable and every compact abelian group is almost metrizable. Further, it is a consequence of the structure theorem for locally compact abelian groups that every locally compact abelian group is almost metrizable.

**Definition 6.15.** A Hausdorff space  $X$  is called **locally  $k_\omega$  space** (when endowed with the subspace topology) if every point has an open neighborhood which is a  $k_\omega$ -space.

Let us first collect some properties of abelian groups which are (locally)  $k_\omega$  spaces. Following [8], we will call them (locally)  $k_\omega$  groups.



**Proposition 6.16.** *An abelian Hausdorff group is a locally  $k_\omega$  group if and only if it has an open  $k_\omega$  subgroup.*

*The dual group of a locally  $k_\omega$  group is Čech complete and conversely, the dual group of a Čech complete group is locally  $k_\omega$ .*

*Proof.* These are (5.3) and (6.1) in [8]. □

**Theorem 6.17.** *Every nuclear locally  $k_\omega$  group is strongly reflexive.*

*Proof.* Let  $G$  be a locally  $k_\omega$  group. According to 6.16, there is an open subgroup  $H$  of  $G$  which is a  $k_\omega$  group and (trivially) nuclear. So 6.12 implies that  $H$  is strongly reflexive. In [5] it has been shown that a group is strongly reflexive if it has an open, strongly reflexive subgroup. □

**Remark 6.18.** A result similar to 6.17 has been stated in (4.6) in [1], unfortunately the proof contains an error: the homomorphism  $(f_n^m) : Q_n \rightarrow Q_m$  need not be an embedding.

6.17 generalizes (7.9) in [8].

**Questions 6.19.** Is every strongly reflexive group a nuclear group?

Is every strongly reflexive group a  $k$ -space?

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